

# A TEXTURE BESTIARY\*

James A. Bryan,<sup>(1)</sup> Sean M. Carroll,<sup>(2)</sup> and Ted Pyne<sup>(3)</sup>

<sup>(1)</sup>*Department of Mathematics, Harvard University  
Cambridge, Massachusetts 02138  
email: gruad@zariski.harvard.edu*

<sup>(2)</sup>*Center for Theoretical Physics, Laboratory for Nuclear Science  
and Department of Physics  
Massachusetts Institute of Technology  
Cambridge, Massachusetts 02139  
email: carroll@marie.mit.edu*

<sup>(3)</sup>*Harvard-Smithsonian Center for Astrophysics  
Cambridge, Massachusetts 02138  
email: pyne@cfa.harvard.edu*

## Abstract

Textures are topologically nontrivial field configurations which can exist in a field theory in which a global symmetry group  $G$  is broken to a subgroup  $H$ , if the third homotopy group  $\pi_3$  of  $G/H$  is nontrivial. We compute this group for a variety of choices of  $G$  and  $H$ , revealing what symmetry breaking patterns can lead to texture. We also comment on the construction of texture configurations in the different models.

Subtitle “From symmetry-breaking patterns  
to topological field configurations” added in journal.

CTP # 2263  
hep-ph/9312254

December 1993, revised June 1994  
Final Version

---

\* This work was supported in part by NASA under Grants no. NAGW-931 and NGT-50850, by the National Science Foundation under grants AST/90-05038 and PHY/92-06867, and by the U.S. Department of Energy (D.O.E.) under contract no. DE-AC02-76ER03069.

## I. Introduction

Longstanding interest in topological defects such as vortices (cosmic strings) and monopoles has recently been joined by interest in textures, field configurations which are everywhere at the minimum of the potential energy of a theory but cannot be smoothly deformed to a constant configuration without leaving the vacuum manifold. The cosmological implications of textures have been extensively studied [1,2,3], as well as their appearance in ordered media [4]. In field theories with higher-derivative interactions, texture configurations can lead to stable solitons such as Skyrmions [5].

Although textures are conceptually distinct from defects, the computations which reveal their existence are entirely analogous; both depend on the topology of the vacuum manifold  $\mathcal{M}$ , the set of minima of the potential energy in the theory under consideration. A topological defect will necessarily exist if, on a certain region of space, the fields take values in  $\mathcal{M}$  such that it is impossible to find a smooth solution over all of space which does not leave  $\mathcal{M}$ . For example, we may be given the value of the field on a circle  $S^1$  in space. If it is impossible to smoothly deform the image of this circle in  $\mathcal{M}$  to a point, then the field must climb out of  $\mathcal{M}$  somewhere inside the circle, and we know that a vortex must pass through the circle. Clearly, such vortices can exist if the first homotopy group  $\pi_1(\mathcal{M})$ , the set of topologically inequivalent maps from  $S^1 \rightarrow \mathcal{M}$ , is nontrivial. Similarly, monopoles are related to  $\pi_2(\mathcal{M})$  (topologically inequivalent maps of two-spheres into  $\mathcal{M}$ ), and domain walls are related to  $\pi_0(\mathcal{M})$  (topologically inequivalent maps of zero-spheres, or simply the number of disconnected pieces into which  $\mathcal{M}$  falls).

Textures, on the other hand, are field configurations which do remain in the vacuum manifold everywhere in space, but are nevertheless topologically distinct from the “global vacuum” in which the field has no gradient energy and lies at the same point of  $\mathcal{M}$  everywhere. (It follows that textures are only important in theories with spontaneously broken *global* symmetries; in a gauge theory, a “texture” configuration is merely a gauge transform of a constant-field configuration.) A region of space with a spherical boundary may be said to contain a texture if the field lies in  $\mathcal{M}$  at every point, takes the same value everywhere on the boundary and is topologically distinct from a constant-field configuration. Since the boundary maps to a single point in field space, such a configuration defines a map from a three-sphere into the vacuum manifold. Thus, the existence of textures is predicated on the existence of topologically nontrivial maps  $S^3 \rightarrow \mathcal{M}$ , which are classified by the third homotopy group  $\pi_3(\mathcal{M})$ . For a theory in which a symmetry group  $G$  is spontaneously broken to a subgroup  $H$ , the vacuum manifold  $M$  is isomorphic to  $G/H$ . Hence,  $\pi_3(\mathcal{M})$  can be calculated once  $G$  and  $H$  are specified.<sup>1</sup>

---

<sup>1</sup> The manifold  $G/H$  depends not only on  $G$  and  $H$ , but on the specific embedding of

It is not difficult to discover symmetry breaking patterns which lead to texture. For example,  $\text{SO}(4)/\text{SO}(3) = S^3$ , and it is well known that  $\pi_3(S^3) = \mathbf{Z}$ . However, it would be interesting to know when any specified field theory predicts texture. In this paper, we undertake the task of computing  $\pi_3(G/H)$  for a variety of symmetry-breaking patterns, thereby constructing a catalogue of when and in what forms texture can appear. By the standards of modern algebraic topology this is not a sophisticated problem; from the point of view of the practicing cosmologist, however, it is useful to know both what the relevant homotopy groups are, and also how field configurations representing textures may be constructed in a given theory. We have therefore endeavored to calculate the topological properties of the vacuum manifolds in such a way that the discussion would lead directly to the examination of specific field configurations.

We would be remiss if we failed to mention that the texture scenario for cosmological structure formation has been dealt blows on both theoretical and observational grounds. On the theoretical front, it has been noticed that symmetry-breaking operators induced by quantum gravity can drastically alter the evolution of would-be textures, rendering them cosmologically impotent [6]. (At the same time, our understanding of Planck-scale physics is insufficient to make incontrovertible statements about such effects.) Observationally, the texture scenario for structure formation, when normalized to the amplitude of microwave background fluctuations observed by COBE, predicts galaxy velocity dispersions somewhat higher than those observed [3]. (Once again, however, our understanding of galaxy-scale physics is also insufficient to make incontrovertible statements at this time.) Finally, it is not clear to what extent the topologically nontrivial nature of textures is relevant; investigations have shown that interesting perturbations can result from scalar field gradients even when  $\pi_3(\mathcal{M})=0$  [7].

Nevertheless, the resilient nature of cosmological models and the possible appearance of texture in other contexts suggests to us that our results are still interesting. Furthermore, in the absence of symmetry-breaking operators, a theory which might predict texture must now be shown to be consistent with the large-scale structure data, and the computations done below help to make this possible.

## II. Basic Techniques

### A. Set-up

In this section we describe some general techniques used to compute  $\pi_3(G/H)$  for

---

$H$  in  $G$ . In the course of the paper we will specify how  $H$  acts as a subgroup of  $G$  for each of the models we consider.

various choices of symmetry breaking patterns. We review some basic homotopy lore, including a description of the exact sequence, and describe some simplifications which occur when  $H$  is abelian.

We first recall the homotopy groups of various spaces, including the Lie groups in which we are interested. This information can be found in reference books, and we have compiled those groups which are relevant to us in Table I. It is also useful to know that the homotopy of a product of two spaces is simply the sum of the individual homotopy groups,

$$\pi_q(X \times Y) = \pi_q(X) \oplus \pi_q(Y). \quad (2.1)$$

Finally, we shall make use of the fact that  $SU(n)/SU(n-1)$  and  $SO(n)/SO(n-1)$  are homeomorphic to spheres:

$$\begin{aligned} SU(n)/SU(n-1) &\sim S^{2n-1} \\ SO(n)/SO(n-1) &\sim S^{n-1}. \end{aligned} \quad (2.2)$$

Notice that, since  $SU(1)$  and  $SO(1)$  are both the trivial group,  $SU(2) \sim S^3$  and  $SO(2) \sim S^1$ . The usefulness of these relations stems from the simple nature of the lower homotopy groups of the spheres:

$$\pi_q(S^n) = \begin{cases} 0 & \text{for } q < n, \\ \mathbf{Z} & \text{for } q = n. \end{cases} \quad (2.3)$$

For  $q > n$  there is no easy relation analogous to (2.3); in fact, the computation of  $\pi_q(S^n)$  is an open problem.

## B. Exact Homotopy Sequence

The exact homotopy sequence is a sequence of maps between homotopy groups of two spaces and those of their quotient space. If  $G$  is a Lie group with subgroup  $H$ , we have

$$\dots \longrightarrow \pi_{q+1}(G/H) \xrightarrow{\gamma_{q+1}} \pi_q(H) \xrightarrow{\alpha_q} \pi_q(G) \xrightarrow{\beta_q} \pi_q(G/H) \xrightarrow{\gamma_q} \pi_{q-1}(H) \longrightarrow \dots \quad (2.4)$$

The maps  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  are specified in terms of the spaces  $G$ ,  $H$  and  $G/H$ , and they are all group homomorphisms. For example, the map  $\alpha_q$  takes the image of a  $q$ -sphere in  $H$  into an image of a  $q$ -sphere in  $G$  using the inclusion of  $H$  as a subgroup in  $G$ ,  $i : H \hookrightarrow G$ . For more details see [8,9].

“Exactness” means that the image of each map is precisely equal to the kernel (the set of elements taken to zero) of the map following it. An important consequence of exactness is that if two spaces  $A$  and  $B$  are sandwiched between the trivial group,  $0 \rightarrow A \xrightarrow{\phi} B \rightarrow 0$ , then the map  $\phi$  must be an isomorphism. This is easy to see: since the kernel of  $B \rightarrow 0$  is all of  $B$ ,  $\phi$  must be onto. Meanwhile, since the kernel of  $\phi$  is the image of  $0 \rightarrow A$  (which is just zero), in order for  $\phi$  to be a group homomorphism it must be one-to-one. Thus,  $\phi$  is an isomorphism.

The exact homotopy sequence can also be used when the space  $G$  appearing in (2.4) is not a Lie group, as long as  $G$  may be thought of as a fiber bundle with fiber  $H$  and base space  $G/H$ . For example, if  $H$  is a group which acts freely (only the identity has fixed points) on a manifold  $M$ , then  $M$  may be thought of as a fiber bundle with base space  $M/H$ . The usefulness of this fact arises when a global symmetry group  $G$  breaks spontaneously to a subgroup of the form  $H_1 \times H_2$ . Then we may think of  $G/(H_1 \times H_2)$  as  $(G/H_1)/H_2$ , even if  $G/H_1$  is not a group, since the action of  $H_2$  divides  $G/H_1$  into well-defined equivalence classes. Thus, we can apply (2.4) with  $G/H_1$  playing the role of  $G$  and  $H_2$  playing the role of  $H$ . It is important, however, not to get carried away; if a subgroup  $H$  of  $G$  can be written in the form  $A/B$ , it is in no way permissible to think of  $G/(A/B)$  as  $(G \times B)/A$ .

In computing  $\pi_3(G/H)$ , we take advantage of the fact that, for any Lie group  $H$ ,  $\pi_2(H) = 0$ . We therefore consider

$$\pi_3(H) \xrightarrow{\alpha} \pi_3(G) \xrightarrow{\beta} \pi_3(G/H) \xrightarrow{\gamma} 0 . \quad (2.5)$$

The map  $\gamma$  takes all of  $\pi_3(G/H)$  to zero. Since the sequence is exact, the image of  $\beta$  is therefore all of  $\pi_3(G/H)$ ; *i.e.*  $\beta$  is onto. This implies that  $\pi_3(G/H)$  is isomorphic to the domain of  $\beta$  (which is all of  $\pi_3(G)$ ) modulo the kernel of  $\beta$  (which we know is  $\text{Im } \alpha$ ). Thus,

$$\pi_3(G/H) \cong \pi_3(G)/\text{Im } \alpha . \quad (2.6)$$

Therefore, our task is reduced to computing  $\pi_3(G)$  (usually easy to do) and  $\text{Im } \alpha$  (sometimes hard to do).

### C. Abelian Subgroups

Consider the case of a general global symmetry group  $G$  spontaneously breaking down to an abelian subgroup  $A$ . All compact, connected, abelian Lie groups are of the form  $A = \text{U}(1) \times \text{U}(1) \times \dots \times \text{U}(1)$ ; disconnected groups will be of this form times various factors of  $\mathbf{Z}$  and  $\mathbf{Z}_n$ . Using the product rule (2.1), along with  $\pi_3(\text{U}(1)) = \pi_3(\mathbf{Z}) = \pi_3(\mathbf{Z}_n) = 0$ , we have  $\pi_3(A) = 0$  for any abelian group  $A$ . (The compactness condition is actually unnecessary.) As Turok [1] has pointed out, the exact sequence

$$\begin{array}{ccccccc} \pi_3(A) & \longrightarrow & \pi_3(G) & \longrightarrow & \pi_3(G/A) & \longrightarrow & \pi_2(A) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array} \quad (2.7)$$

then implies that

$$\pi_3(G/A) = \pi_3(G), \quad A \text{ abelian} . \quad (2.8)$$

This includes, of course, the case  $A = 0$ . Further, when a subgroup  $H$  can be decomposed into  $H = K \times A$ , with  $K$  an arbitrary Lie group and  $A$  abelian, we can use the reasoning of Sec. II.B to show that

$$\pi_3(G/(K \times A)) = \pi_3(G/K), \quad A \text{ abelian} . \quad (2.9)$$

The interesting examples in which  $H$  is non-abelian must be handled on a case-by-case basis.

## III. Simple Groups

### A. General Procedure

In this section we present a formula which can be combined with (2.6) to compute  $\pi_3(G/H)$  when  $G$  and  $H$  are both simple.<sup>2</sup> (Generalization to non-simple groups is straightforward.) The important property of simple groups is that  $\pi_3(G) = \mathbf{Z}$  for any

---

<sup>2</sup> We are very grateful to Sidney Coleman for explaining to us the procedure outlined in this section.

simple group  $G$ ; the element of  $\mathbf{Z}$  to which a map  $g : S^3 \rightarrow G$  corresponds is called the winding number of  $g$ , or  $\eta_G(g)$ . (We denote the range space of  $g$  as a subscript for clarity.) The map  $\alpha : \pi_3(H) \rightarrow \pi_3(G)$  whose image we wish to compute is a homomorphism from  $\mathbf{Z}$  to  $\mathbf{Z}$ , which may be thought of as multiplication by an integer  $p$ . In other words, a map  $f : S^3 \rightarrow H$  with winding number  $\eta_H(f)$  will, when composed with the inclusion map  $i : H \hookrightarrow G$ , yield a map with winding number

$$\eta_G(i \circ f) = p\eta_H(f) . \quad (3.1)$$

Hence, the image of  $\alpha$  is every  $p$ th integer, so (2.6) and  $\pi_3(G) = \mathbf{Z}$  together imply that

$$\pi_3(G/H) = \mathbf{Z}_p , \quad (3.2)$$

the group of integers modulo addition by  $p$ . In the case  $p = 0$  this is interpreted as  $\mathbf{Z}_0 = \mathbf{Z}$ , while  $\mathbf{Z}_1$  is the trivial group. Our aim is therefore to compute  $p$ .

Clearly, it should be possible to compute  $p$  by choosing a generator  $f$  of  $\pi_3(H)$  (a map with unit winding number,  $\eta_H(f) = 1$ ); then (3.1) implies that  $p$  is equal to the winding number of  $i \circ f$  in  $G$ . Rather than computing this winding number directly, we shall define a number  $\xi_G(\phi)$  which is proportional to the winding number of a map  $\phi$ , and then take the ratio of  $\xi_G(i \circ f)$  to the equivalent expression for a map with winding number one. In other words, since  $p = \eta_G(i \circ f) \propto \xi_G(i \circ f)$  for a map  $f$  which generates  $\pi_3(H)$ , we shall choose a map  $g : S^3 \rightarrow G$  which generates  $\pi_3(G)$  (thus,  $\eta_G(g) = 1$ ), and calculate the ratio

$$p = \frac{\xi_G(i \circ f)}{\xi_G(g)} . \quad (3.3)$$

To do so, we take advantage of a remarkable fact unique to  $\pi_3$  (as opposed to the other  $\pi_q$ ). Given a map  $\phi : S^3 \rightarrow G$  which takes a point labeled by  $x$  to a matrix  $\phi(x)$ , there is an integral expression proportional to the winding number:

$$\eta_G(\phi) \propto \int_{S^3} d^3x \epsilon^{ijk} \text{Tr} [(\partial_i \phi) \phi^{-1} (\partial_j \phi) \phi^{-1} (\partial_k \phi) \phi^{-1}] , \quad (3.4)$$

where matrix multiplication is implicit. A demonstration of the topological invariance of this quantity can be found in [10]. Thus,  $p$  will follow from Eq. (3.3) if we can evaluate (3.4) for the maps  $i \circ f$  and  $g$ .

Fortunately it is not even necessary to evaluate (3.4), as we can use another remarkable fact unique to  $\pi_3$ . This is a theorem due to Bott [11], that a map  $g$  representing the generator of  $\pi_3(G)$  may always be taken to be a homomorphism from  $SU(2)$  to  $G$ , while maps with winding number  $n$  may be represented by  $\phi = g^n$ . The integral in (3.4) will then be equal to a constant factor (proportional to the volume of  $SU(2)$ , but independent of  $\phi$ ) times the value of the integrand at any point. We may take this point to be the identity element of  $SU(2)$  (with coordinates  $x = 0$ ), which is mapped to the identity element  $e$  in  $G$ . Thus, (3.4) is equivalent to

$$\eta_G(\phi) \propto \epsilon^{ijk} \text{Tr} [\partial_i \phi(x) \partial_j \phi(x) \partial_k \phi(x)] \big|_{x=0} , \quad (3.5)$$

where we have used  $\phi(0)^{-1} = e^{-1} = e$ . Now, the quantity  $(\partial_i \phi(x)) \phi^{-1}(x)$  evaluated at  $x = 0$  defines an element of the Lie algebra of  $G$ . Since the group map  $\phi$  defines an algebra map

$$\begin{aligned} \phi_* : \mathcal{SU}(2) &\rightarrow \mathcal{G} \\ X_i &\mapsto \tilde{X}_i , \end{aligned} \quad (3.6)$$

(3.5) takes the form

$$\eta_G(\phi) \propto \epsilon^{ijk} \text{Tr} \left( \tilde{X}_i \tilde{X}_j \tilde{X}_k \right) . \quad (3.7)$$

The matrices  $\tilde{X}_i$  are elements of  $\mathcal{G}$ , the Lie algebra of  $G$ , but by themselves they form a basis for  $\mathcal{SU}(2)$ , the Lie algebra of  $SU(2)$ . Specifically, they satisfy commutation relations

$$[\tilde{X}_i, \tilde{X}_j] = n \epsilon^{ijk} \tilde{X}_k , \quad (3.8)$$

from which it follows immediately that

$$\epsilon^{ijk} \tilde{X}_i \tilde{X}_j = 2n \tilde{X}_k , \quad (3.9)$$



where  $n$  is the winding number of the map  $\phi$ . (If  $\phi$  is the generator of  $\pi_3(G)$ , we have  $n = 1$ , and  $\phi_*$  is a Lie algebra homomorphism.) Thus, (3.7) becomes

$$\eta_G(\phi) \propto \text{Tr} \left[ (\tilde{X}_1)^2 + (\tilde{X}_2)^2 + (\tilde{X}_3)^2 \right] . \quad (3.10)$$

Since the three  $\tilde{X}_i$  are a basis for  $\mathcal{SU}(2)$  (in some representation), the quantity  $\text{Tr}(\tilde{X}_i)^2$  is the same for each  $i$ . If we therefore pick, for example,  $i = 3$ , we may say

$$\eta_G(\phi) \propto \text{Tr}(\tilde{X}_3)^2 . \quad (3.11)$$

We may notice a resemblance between (3.10) and the expression for the second Casimir operator of the representation. For an irreducible  $d$ -dimensional representation  $\Lambda$ , this is a matrix

$$\begin{aligned} C_2(\Lambda) &= (\Lambda_1)^2 + (\Lambda_2)^2 + (\Lambda_3)^2 \\ &= J(J+1)\mathbf{I}_d , \end{aligned} \quad (3.12)$$

where the  $\Lambda_i$  are the basis for  $\mathcal{SU}(2)$  in this representation,  $\mathbf{I}_d$  is a  $d \times d$  identity matrix, and  $J$  is the spin of the representation (related to  $d$  by  $J = (d-1)/2$ ). Thus, if the representation  $\tilde{X}$  is irreducible, the right hand side of (3.10) is just  $\text{Tr} C_2(\tilde{X})$ ; if it is reducible, it will be a sum of  $\text{Tr} C_2(\Lambda^A)$  for each irreducible factor  $\Lambda^A$  in  $\tilde{X}$ . The trace simply introduces a factor of  $d$ ; hence, (3.10) becomes

$$\eta_G(\phi) \propto \sum_{\text{irreps } A} d_A(d_A - 1)(d_A + 1) . \quad (3.13)$$

(Remember that the proportionality sign means that the winding number equals the expression on the right hand side times constant factors which do not depend on the map  $\phi$ .) Notice that (3.13) allows us to determine what maps  $\phi$  determine the generator of  $\pi_3(G)$ ; in principle, we can examine all homomorphisms  $\mathcal{SU}(2) \rightarrow \mathcal{G}$ , and search for the one for which (3.13) is the lowest. In practice, it is usually easy to find the generator.

Since the right hand sides of (3.11) and (3.13) are each proportional to  $\eta_G(\phi)$ , they clearly must be proportional to each other. It will be convenient to make this proportionality explicit for a certain choice of conventions. Thus, we will define

$$\begin{aligned}\xi_G(\phi_*) &= -2 \text{Tr}(\tilde{X}_3)^2 \\ &= \frac{1}{6} \sum_{\text{irreps } A} d_A(d_A - 1)(d_A + 1) ,\end{aligned}\tag{3.14}$$

a number characterizing the algebra map  $\phi_* : \mathcal{SU}(2) \rightarrow \mathcal{G}$  (and the associated group map  $\phi$ ). Note that the expression  $\text{Tr}(\tilde{X}_3)^2$  is indeed convention dependent; thus, (3.14) represents a statement about the conventions we shall choose below. Note also that the factor  $1/6$  sets  $\xi_G(\phi_*) = 1$  when the representation of  $\mathcal{SU}(2)$  induced by  $\phi_*$  is two-dimensional and irreducible; this is the lowest possible value for maps which define nontrivial representations. Finally, it is important to distinguish between  $\xi_G(\phi_*)$  and the winding number  $\eta_G(\phi)$ ; there is no guarantee that  $\xi_G(g_*)$  will be unity for a map  $g : \text{SU}(2) \rightarrow G$  which generates  $\pi_3(G)$ , so we must normalize the winding number appropriately:

$$\eta_G(\phi) = \frac{\xi_G(\phi_*)}{\xi_G(g_*)} .\tag{3.15}$$

Therefore, in order to compute  $p$  from (3.3), we need to choose homomorphisms  $f : \text{SU}(2) \rightarrow H$  and  $g : \text{SU}(2) \rightarrow G$ , which are generators of  $\pi_3$  in their respective groups ( $\eta_H(f) = 1$  and  $\eta_G(g) = 1$ ).<sup>3</sup> We then find the associated maps (3.6) between the Lie algebras, and take the ratio of (3.14) evaluated for  $i \circ f$  and  $g$ . In Sec. IV we shall go through this procedure for various symmetry-breaking patterns.

## B. Finding the Generators

In order to apply the procedure just outlined, it is necessary to find maps from  $\text{SU}(2)$  to both  $G$  and  $H$  which generate the third homotopy group. As mentioned previously, it is sufficient to find a homomorphism between the Lie algebra of  $\text{SU}(2)$  and that of the group; this will define the group homomorphism. We now set about finding the relevant maps for the cases of interest to us, namely  $\text{SU}(n)$ ,  $\text{SO}(n)$  and the symplectic groups  $\text{Sp}(n)$ .

We begin by considering  $\text{SU}(2)$ . Since  $\text{SU}(2)$  is homeomorphic to  $S^3$ , and  $\pi_3(S^3)$  is generated by the identity map,  $\pi_3(\text{SU}(2))$  is generated by the identity homomorphism.

---

<sup>3</sup> As  $i : H \rightarrow G$  is already a homomorphism, we need only to choose  $f$  to be a homomorphism  $\text{SU}(2) \rightarrow H$  for  $i \circ f$  to be a homomorphism.

Let us think about this fact on the Lie algebra level. The usual convention in physics is to define a Lie algebra element  $T$  of  $SU(n)$  to be traceless and Hermitian, such that group elements are of the form  $g(T) = \exp(iT)$ . For convenience we shall use the notation common in the mathematics literature, where the Lie algebra consists of matrices  $X$  which are traceless and skew-Hermitian ( $X^\dagger = -X$ ), such that group elements are of the form  $g(X) = \exp(X)$ . Thus, we take the basis vectors  $X_i$  of  $SU(2)$  in the defining representation to be

$$X_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad X_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad (3.16)$$

with commutation relations

$$[X_i, X_j] = \epsilon^{ijk} X_k . \quad (3.17)$$

As this is an irreducible two-dimensional representation, the (trivial) Lie algebra homomorphism  $g_* : X_i \mapsto X_i$  has

$$\xi_{SU(2)}(g_*) = 1 . \quad (3.18)$$

(Note that, in the conventions we have chosen, both expressions in (3.14) yield the same value for  $\xi_{SU(2)}(g_*)$ .) This is the lowest possible value that  $\xi$  can take for any map, and hence the identity homomorphism  $g : SU(2) \rightarrow SU(2)$  specified by  $g_*$  is seen to be the generator of  $\pi_3(SU(2))$ , in accordance with expectation.

For all  $SU(n)$   $n \geq 2$ , we have  $\pi_3(SU(n)) = \mathbf{Z}$ . It is easy to find a map which generates  $\pi_3$ , since there is a natural map  $g_* : SU(2) \rightarrow SU(n)$ , given by placing the  $2 \times 2$  matrices representing  $X_i$  in the upper left hand corner of  $n \times n$  matrices which are zero elsewhere:

$$g_* : X_i \mapsto \tilde{X}_i \equiv \begin{pmatrix} (X_i) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} . \quad (3.19)$$

The Lie algebra elements  $\tilde{X}_i$ , upon exponentiation, define an  $SU(2)$  subgroup of  $SU(n)$  with an element of  $SU(2)$  in the upper left hand corner, ones on the rest of the diagonal,

and zeroes elsewhere. We therefore may extend (3.18) to

$$\xi_{\text{SU}(n)}(g_*) = 1, \quad n \geq 2. \quad (3.20)$$

Once again, as this is the minimum value  $\xi$  may take, the group homomorphism  $g : \text{SU}(2) \rightarrow \text{SU}(n)$  specified by (3.19) is a generator of  $\pi_3(\text{SU}(n))$ .

The case of  $\text{SO}(n)$  is somewhat more interesting.  $\text{SO}(2)$  is topologically a circle, and thus  $\pi_3(\text{SO}(2)) = 0$ . In the case of  $\text{SO}(3)$ , on the other hand, the Lie algebra is isomorphic to that of  $\text{SU}(2)$ , and this isomorphism specifies a map  $g : \text{SU}(2) \rightarrow \text{SO}(3)$  which generates  $\pi_3(\text{SO}(3))$ . This map is the familiar double cover of  $\text{SO}(3)$  by  $\text{SU}(2)$ ; indeed, we may think of  $\text{SO}(3)$  as  $\text{SU}(2)/\mathbf{Z}_2$ , in which case the exact homotopy sequence

$$\begin{array}{ccccccc} \pi_3(\mathbf{Z}_2) & \longrightarrow & \pi_3(\text{SU}(2)) & \xrightarrow{\beta} & \pi_3(\text{SO}(3)) & \longrightarrow & \pi_2(\mathbf{Z}_2) \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbf{Z} & & \mathbf{Z} & & 0 \end{array} \quad (3.21)$$

tells us immediately that the map  $\beta$  is an isomorphism. Thus, the generator of  $\pi_3(\text{SO}(3))$  is a map from the three-sphere into  $\text{SO}(3)$  which wraps twice around the group. Let's see how this appears at the Lie algebra level. The Lie algebra of  $\text{SO}(n)$  is given by antisymmetric real  $n \times n$  matrices. For  $n = 3$ , we may choose as a basis three matrices  $Y_i$ , given by

$$Y_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.22)$$

which have commutation relations

$$[Y_i, Y_j] = \epsilon^{ijk} Y_k. \quad (3.23)$$

The isomorphism  $g_* : \mathcal{SU}(2) \rightarrow \mathcal{SO}(3)$  is therefore immediate:

$$g_* : X_i \mapsto Y_i. \quad (3.24)$$

This isomorphism establishes the matrices (3.22) as a representation of  $\mathcal{SU}(2)$ ; it is three-dimensional and irreducible. Thus,

$$\xi_{\text{SO}(3)}(g_*) = 4. \quad (3.25)$$

We can be certain that the group homomorphism defined by  $g_*$  does generate  $\pi_3(\text{SO}(3))$ , as there are no two-dimensional representations of  $\mathcal{SU}(2)$  by real antisymmetric  $3 \times 3$  matrices. Therefore no other homomorphism  $\mathcal{SU}(2) \rightarrow \mathcal{SO}(3)$  will yield a lower value for  $\xi$ .

$\text{SO}(4)$  is something of a special case, in that it is not a simple group. The Lie algebra  $\mathcal{SO}(4)$  is isomorphic to  $\mathcal{SU}(2) \oplus \mathcal{SU}(2)$ , and  $\text{SO}(4)$  itself is isomorphic to  $(\text{SU}(2) \times \text{SU}(2))/\mathbf{Z}_2$ . This latter fact allows us to use the exact homotopy sequence to find  $\pi_3(\text{SO}(4))$ :

$$\begin{array}{ccccccc} \pi_3(\mathbf{Z}_2) & \longrightarrow & \pi_3(\text{SU}(2) \times \text{SU}(2)) & \xrightarrow{\beta} & \pi_3(\text{SO}(4)) & \longrightarrow & \pi_2(\mathbf{Z}_2) \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbf{Z} \oplus \mathbf{Z} & & \mathbf{Z} \oplus \mathbf{Z} & & 0 \end{array}. \quad (3.26)$$

Thus, there are two generators of  $\pi_3(\text{SO}(4))$ , each of which may be thought of as inherited from a projection map  $p : \text{SU}(2) \times \text{SU}(2) \rightarrow \text{SO}(4)$ . These generators may be specified by exhibiting the Lie algebra isomorphism  $p_* : \mathcal{SU}(2) \oplus \mathcal{SU}(2) \rightarrow \mathcal{SO}(4)$ . A basis for  $\mathcal{SU}(2) \oplus \mathcal{SU}(2)$  is given by  $4 \times 4$  matrices  $X_i^a, X_i^b, i = 1, 2, 3$ , where the  $X_i^a$  consist of the  $\mathcal{SU}(2)$  generators  $X_i$  from (3.16) in the upper left  $2 \times 2$  block and zeroes elsewhere, while the  $X_i^b$  have the  $X_i$  in the lower right block and zeroes elsewhere. As a basis for  $\mathcal{SO}(4)$ , we choose

$$\begin{aligned} Z_1 &= \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & Z_2 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & Z_3 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ Z_4 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & Z_5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & Z_6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.27)$$

It is easy to check that the map  $p_* : X_i^a \mapsto Z_i, X_i^b \mapsto Z_{i+3}$  is an algebra isomorphism. Thus, the submanifolds defined by exponentiation of the two subalgebras  $(Z_1, Z_2, Z_3)$  and  $(Z_4, Z_5, Z_6)$  serve as the two generators of  $\pi_3(\text{SO}(4))$ . As representations of  $\mathcal{SU}(2)$ , these

subalgebras are each reducible into a sum of two two-dimensional representations. We can evaluate (3.14) for the homomorphism  $p_*$  restricted to either the  $X_i^a$  or the  $X_i^b$ , obtaining

$$\xi_{\text{SO}(4)}[p_*(X_i^a)] = \xi_{\text{SO}(4)}[p_*(X_i^b)] = 2 . \quad (3.28)$$

Notice that this number is lower than the result (3.25) for  $\text{SO}(3)$ . This is because the representation of  $\mathcal{SU}(2)$  given by  $(Z_1, Z_2, Z_3)$  cannot fit inside  $\mathcal{SO}(3)$ ; the sum of two irreducible two-dimensional representations cannot be expressed as real  $3 \times 3$  antisymmetric matrices. With  $4 \times 4$  matrices there is no difficulty, and we are able to find a representation for which  $\xi$  is lower.

For  $n > 4$ , the  $\text{SO}(n)$  are all simple, and hence  $\pi_3(\text{SO}(n)) = \mathbf{Z}$ . We may choose the first six basis elements for  $\mathcal{SO}(n)$  to be  $n \times n$  matrices with the matrices  $Z_i$  from (3.27) in the upper left corner and zeroes elsewhere. (We will refer to the resulting matrices also as  $Z_i$ ; no confusion should arise.) A homomorphism  $\mathcal{SU}(2) \rightarrow \mathcal{SO}(n)$  is given by

$$g_* : X_i \mapsto Z_i , \quad i = 1, 2, 3 . \quad (3.29)$$

Just as in (3.28), this representation has  $\xi_{\text{SO}(n)}(g_*) = 2$ , which guarantees that the group homomorphism  $g$  defined by the algebra homomorphism  $g_*$  is a generator of  $\pi_3(\text{SO}(n))$ , as there is no two-dimensional representation of  $\mathcal{SU}(2)$  via real antisymmetric  $n \times n$  matrices. On the other hand, we could also consider the homomorphism  $g'_* : X_i \mapsto Z_{i+3}$ ,  $i = 1, 2, 3$ . This also defines a generator of  $\pi_3(\text{SO}(n))$ ; however, unlike in  $\text{SO}(4)$ , in  $\text{SO}(n > 4)$  these two generators are actually homotopic to each other. Therefore we may take either one as the generator, and we shall generally choose (3.29).

Finally we consider the symplectic groups  $\text{Sp}(2n)$ , all of which are simple.<sup>4</sup> As basis elements for the algebra  $\mathcal{SP}(2n)$  we may choose

$$\mathbf{I}_2 \otimes A_j^{(n)} , \quad X_i \otimes S_k^{(n)} , \quad (3.30)$$

where  $\mathbf{I}_2$  is a  $2 \times 2$  identity matrix, the  $X_i$  are the  $2 \times 2$  elements of  $\mathcal{SU}(2)$  defined in (3.16), the  $A_j^{(n)}$  ( $j = 1, \dots, \frac{1}{2}n(n-1)$ ) are a basis for real antisymmetric  $n \times n$  matrices,

---

<sup>4</sup> We are taking  $\text{Sp}(2n)$  to be the symplectic group consisting of  $2n \times 2n$  matrices, with  $n$  an integer. Other conventions denote this same group by  $\text{Sp}(n)$ .

and the  $S_k^{(n)}$  ( $k = 1, \dots, \frac{1}{2}n(n+1)$ ) are a basis for real symmetric  $n \times n$  matrices. Note that all of these matrices are traceless and skew-Hermitian; hence,  $\mathcal{SP}(2n)$  is naturally a subalgebra of  $\mathcal{SU}(2n)$ . Let  $S_1^{(n)}$  be the  $n \times n$  matrix with a 1 in the upper left corner and zeroes elsewhere. Then there is a homomorphism  $g_* : \mathcal{SU}(2) \rightarrow \mathcal{SP}(2n)$  given by

$$g_* : X_i \rightarrow X_i \otimes S_1^{(n)} . \quad (3.31)$$

For this (two-dimensional) representation we have

$$\xi_{\text{Sp}(2n)}(g_*) = 1 . \quad (3.32)$$

As this is the lowest value possible, the group homomorphism specified by (3.31) must be a generator of  $\pi_3(\text{Sp}(2n))$ .

#### IV. Examples

In this section we apply the general formulae just derived to explicit computation of  $\pi_3(G/H)$  for different choices of  $G$  and  $H$ . We shall consider the cases  $G = \text{SU}(n)$  and  $G = \text{SO}(n)$ . For these groups, Li [12] has calculated what subgroup  $H$  remains unbroken when a set of scalar fields  $\phi$  transforming under a specified representation of  $G$  attains a specified vacuum expectation value. In the interest of completeness we also compute  $\pi_1(G/H)$  and  $\pi_2(G/H)$ .

It is important to note that a theory invariant under global  $\text{SU}(n)$  or  $\text{SO}(n)$  transformations is sometimes invariant under the larger symmetry groups  $\text{U}(n)$  or  $\text{O}(n)$ , respectively. The existence of this extra degree of symmetry may affect the corresponding vacuum manifold, if the extra symmetry is broken by a vacuum expectation value. In what follows we will assume that the true symmetry really is  $\text{SU}(n)$  or  $\text{SO}(n)$ ; the generalization is straightforward. In any event, only the first and second homotopy groups of the vacuum manifold can depend on this distinction, and therefore the prediction of textures is unaltered.

##### A. $G = \text{SU}(n)$

Consider a set of scalar fields  $\phi^a$  ( $a = 1 \dots n$ ), which transform in the vector representation of a global symmetry group  $\text{SU}(n)$ :

$$\phi^a \rightarrow \phi^{a'} = U^{a'}{}_a \phi^a , \quad (4.1)$$

where  $U$  is an  $n \times n$  matrix representing an element of  $SU(n)$ . Now imagine that  $\phi^a$  attains a vacuum expectation value

$$\langle \phi^a \rangle = \rho \delta_{an} , \quad (4.2)$$

where  $\rho$  is a constant parameter and  $\delta$  is the Kronecker delta. The unbroken subgroup consists of those transformations which leave (4.2) invariant; in this case,  $H = SU(n-1)$ , as explained in Li [12]. We therefore wish to compute  $\pi_3(SU(n)/SU(n-1))$ . Our task is facilitated by the fact that  $SU(n)/SU(n-1)$  is homeomorphic to  $S^{2n-1}$  (Eq. 2.1). We are only interested in  $n \geq 2$  (since  $SU(0)$  does not exist), and hence in spheres  $S^q$  with  $q \geq 3$ . In this case  $\pi_3$  follows immediately from (2.2):

$$\pi_3(SU(n)/SU(n-1)) = \begin{cases} \mathbf{Z} & \text{for } n = 2 , \\ 0 & \text{for } n > 2 . \end{cases} \quad (4.3)$$

We can derive this same result using the techniques of Sec. III. There we found that the generator of  $\pi_3(SU(n))$  could be taken to be an  $SU(2)$  subgroup in the upper left corner of the  $SU(n)$  matrices. Since the subgroup  $H = SU(n-1)$  consists of  $(n-1) \times (n-1)$  matrices in the upper left corner, it is clear that (for  $n \geq 3$ ) the map  $g : SU(2) \rightarrow SU(n)$  generating  $\pi_3(SU(n))$  is precisely the same as the composition  $i \circ f : SU(2) \rightarrow SU(n)$ , where  $f$  is the generator of  $\pi_3(SU(n-1))$  and  $i : SU(n-1) \hookrightarrow SU(n)$  is the inclusion of  $H$  in  $G$ . Therefore (3.3) becomes

$$p = \frac{\xi_{SU(n)}(i \circ f)}{\xi_{SU(n)}(g)} = 1 . \quad (4.4)$$

According to (3.2),  $p$  serves to determine  $\pi_3(G/H) = \mathbf{Z}_p$ ; hence,  $\pi_3(SU(n)/SU(n-1)) = \mathbf{Z}_1 = 0$  when  $n \geq 3$ . (“ $\mathbf{Z}_1$ ,” the integers modulo division by 1, is the trivial group.) Thus (4.3) is verified.

We next consider the case of  $l$  distinct  $n$ -vectors  $\phi_j^a$  ( $j = 1 \dots l$ ), all of which acquire linearly independent vacuum expectation values. We may take all of the nonzero components of  $\langle \phi_j^a \rangle$  to be  $a = n-l+1, n-l, \dots, n$ . The original  $SU(n)$  symmetry is broken to  $H = SU(n-l)$ . Once again, for  $n-l \geq 2$ , the maps  $f$  (generating  $\pi_3(H)$ ) and  $g$  (generating  $\pi_3(G)$ ) satisfy  $i \circ f = g$ , where  $i$  is the inclusion of  $H$  in  $G$ . Thus, (4.4) holds true for this case as well, and we obtain

$$\pi_3(SU(n)/SU(n-l)) = 0 , \quad n-l \geq 2 . \quad (4.5)$$



The same result may be obtained by the factorization procedure of the Appendix, by considering the map  $i : \text{SU}(n-l) \rightarrow \text{SU}(n)$  as the composition  $i_{n-1} \circ i_{n-2} \circ \dots \circ i_{n-l}$ , where  $i_m : \text{SU}(m) \hookrightarrow \text{SU}(m+1)$  is the usual inclusion into the upper left corner. Then the map between homotopy groups  $\alpha : \pi_3(\text{SU}(n-l)) \rightarrow \pi_3(\text{SU}(n))$  factors similarly; however, since  $\pi_3(\text{SU}(m+1)/\text{SU}(m)) = 0$  (for  $m \geq 2$ ), each  $\alpha_m : \pi_3(\text{SU}(m)) \rightarrow \pi_3(\text{SU}(m+1))$  must be an isomorphism (by Eq. (2.6)). Thus  $\alpha$  itself is an isomorphism, and (4.5) is recovered.

We now consider scalar fields transforming in the symmetric second-rank tensor representation,<sup>5</sup>

$$\phi_{a'b'} = U^a_{a'} U^b_{b'} \phi_{ab} , \quad (4.6)$$

where  $\phi_{ab} = \phi_{ba}$ . Such a fields may attain a vacuum expectation value

$$\langle \phi_{ab} \rangle = \rho \delta_{ab} . \quad (4.7)$$

This serves to break  $\text{SU}(n)$  to  $\text{SU}(n-1)$ , and the analysis is identical to the vector case, culminating in (4.3). However, a different choice of potential  $V(\phi_{ab})$  can lead to a vacuum expectation value

$$\langle \phi_{ab} \rangle = \rho \delta_{ab} , \quad (4.8)$$

with  $\rho$  once again a constant [12]. In this case the unbroken symmetry group is  $H = \text{SO}(n)$ ; note that  $\text{SO}(n)$  is naturally a subgroup of  $\text{SU}(n)$ , as all real orthogonal matrices are automatically unitary. The appearance of monopoles in this model was studied in Ref. [13]. We can proceed to compute  $\pi_3(\text{SU}(n)/\text{SO}(n))$  on a case-by-case basis. For  $n = 2$ ,  $\text{SO}(2)$  is abelian, and hence  $\pi_3(\text{SU}(2)/\text{SO}(2)) = \pi_3(\text{SU}(2)) = \mathbf{Z}$ . For  $n = 3$ , we recall from Sec. II.B that the generator  $f$  of  $\pi_3(\text{SO}(3))$  is a double cover, with  $\xi_{\text{SO}(3)}(f_*) = 4$  (3.25). Since the inclusion  $i : \text{SO}(3) \hookrightarrow \text{SU}(3)$  is trivial, we will also have  $\xi_{\text{SU}(3)}[(i \circ f)_*] = 4$ . Meanwhile, the generator  $g$  of  $\pi_3(\text{SU}(3))$  has  $\xi_{\text{SU}(3)}(g_*) = 1$ . It then follows from (3.14) that the

---

<sup>5</sup> Fields transforming in the symmetric or antisymmetric tensor representations may be invariant under an extra degree of symmetry, making the full symmetry group  $U(n)$  rather than  $\text{SU}(n)$ . This may affect  $\pi_1$  of the vacuum manifold, but will not affect  $\pi_3$ , and we will not consider it in this paper.

winding number of  $i \circ f$  in  $G$  is 4, from which we obtain  $\pi_3(\text{SU}(3)/\text{SO}(3)) = \mathbf{Z}_4$ . The case  $n \geq 5$  is equally straightforward; the only difference is that in this case  $\xi_{\text{SO}(n)}(f_*) = 2$ , leading to  $\pi_3(\text{SU}(n)/\text{SO}(n)) = \mathbf{Z}_2$ ,  $n \geq 5$ . The only subtle case is  $n = 4$ , due to the fact that  $\text{SO}(4)$  is not simple, and  $\pi_3(\text{SO}(4)) = \mathbf{Z} \oplus \mathbf{Z}$ . Nevertheless, the logic leading up to (2.6) is still valid, so we need only to find the image of  $\alpha : \pi_3(\text{SO}(4)) \rightarrow \pi_3(\text{SU}(4))$ . As this is a homomorphism from  $\mathbf{Z} \oplus \mathbf{Z}$  to  $\mathbf{Z}$ , it must be of the form  $\alpha : (a, b) \mapsto q_a a + q_b b$  for some integers  $q_a, q_b$ . We have already discovered in Sec. III.B that the generators of  $\pi_3(\text{SO}(4))$  (corresponding to  $(a, b) = (1, 0)$  and  $(0, 1)$ ) may be obtained by exponentiating the subalgebras  $(Z_1, Z_2, Z_3)$  and  $(Z_4, Z_5, Z_6)$  as specified in (3.27), and furthermore that (3.14) evaluated on either generator gives  $\xi_{\text{SO}(4)}(f_*) = 2$ , where  $f_*$  is taken to be the map from  $\text{SU}(2)$  to either generator. Thus, we derive  $q_a = q_b = 2$ . The image of  $\alpha$  is therefore the even numbers; from (2.6) we find  $\pi_3(\text{SU}(4)/\text{SO}(4)) = \mathbf{Z}_2$ . Taken together, we have found

$$\pi_3(\text{SU}(n)/\text{SO}(n)) = \begin{cases} \mathbf{Z} , & n = 2 \\ \mathbf{Z}_4 , & n = 3 \\ \mathbf{Z}_2 , & n \geq 4 . \end{cases} \quad (4.9)$$

We therefore have the interesting situation where  $\pi_3$  can be a finite group, and two textures with positive winding number can mutually unwind rather than collapsing. We comment briefly on this later.

We turn next to the antisymmetric second-rank tensor representation,

$$\phi_{a'b'} = U^a_{a'} U^b_{b'} \phi_{ab} , \quad (4.10)$$

with  $\phi_{ab} = -\phi_{ba}$ . There are two types of vacuum expectation value which can be attained. The first, for which

$$\langle \phi_{ab} \rangle = \rho \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & & \ddots \\ & & & & 0 \end{pmatrix} , \quad (4.11)$$

breaks  $SU(n)$  to  $H = SU(n-2) \times SU(2)$ ; thinking of  $G/H$  as  $[SU(n)/SU(n-l)]/SU(2)$ , it follows from (4.5) that  $\pi_3(G/H) = 0$ . The other alternative is

$$\langle \phi_{ab} \rangle = \rho \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \quad (4.12)$$

when  $n = 2k$  is even, and

$$\langle \phi_{ab} \rangle = \rho \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ 0 & & & & 0 \end{pmatrix} \quad (4.13)$$

when  $n = 2k + 1$  is odd. This pattern breaks  $SU(2k)$  or  $SU(2k + 1)$  to  $H = Sp(2k)$ . As with the orthogonal groups, the symplectic matrices are automatically unitary, so the inclusion  $i : Sp(2k) \hookrightarrow SU(2k+1)$  is trivial. Since we found in Sec. III that the generator  $f$  of  $\pi_3(Sp(2k))$  satisfied  $\xi_{Sp(2k)}(f_*) = 1$ , we must also have  $\xi_{Sp(2k)}[(i \circ f)_*] = 1$ . Hence the winding number of  $i \circ f$  is one, which implies

$$\pi_3(SU(2k+1)/Sp(2k)) = 0, \quad k \geq 1. \quad (4.14)$$

This symmetry breaking pattern therefore does not lead to texture.

The final representation of  $SU(n)$  we consider is the adjoint representation, which transforms as

$$\phi^{a'}_{b'} \rightarrow U^{\dagger a'}_a U^b_{b'} \phi^a_b, \quad (4.15)$$

where  $\phi^a_b$  is an  $n \times n$  skew-Hermitian matrix, and the dagger denotes Hermitian conjugation. The scalar fields  $\phi^a_b$  can attain a vacuum expectation value of the form

$$\langle \phi^a_b \rangle = \begin{pmatrix} \rho_1 & & & & \\ & \rho_1 & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \rho_2 \\ & & & & & \rho_2 \end{pmatrix}, \quad (4.16)$$

where  $\rho_1$  is a constant appearing  $l$  times, and  $\rho_2$  is a constant appearing  $n-l$  times. (By convention we can choose  $n-l \geq l$ .) The result is to break  $SU(n)$  to  $H = S[U(n-l) \times U(l)] = [SU(n-l) \times SU(l) \times U(1)]/\mathbf{Z}_{2(n-l)}$ , leading to vacuum manifolds known as Grassmann spaces. This symmetry breaking pattern is well-known from grand unified theories, where  $SU(5)$  is often said to break to  $SU(3) \times SU(2) \times U(1)$ . While this is true at the Lie algebra level, the unbroken subgroup is actually  $S[U(n-l) \times U(l)]$ , consisting of matrices of the form

$$M = \begin{pmatrix} A & 0_{l \times (n-l)} \\ 0_{(n-l) \times l} & B \end{pmatrix}, \quad \det M = 1, \quad (4.17)$$

where  $A \in U(n-l)$ ,  $B \in U(l)$ , and  $0_{l \times (n-l)}$  is an  $l \times (n-l)$  zero matrix. When  $n-l = l = 1$ , we have  $S[U(1) \times U(1)] = U(1)$ , and hence  $\pi_3(S[U(1) \times U(1)]) = 0$ . For  $n-l > l = 1$ , we have  $\pi_3(S[U(n-l) \times U(1)]) = \mathbf{Z}$ , and for  $n-l \geq l \geq 2$  we have  $\pi_3(S[U(n-l) \times U(l)]) = \mathbf{Z} \oplus \mathbf{Z}$ . In either of the latter cases, one generator of  $\pi_3$  is given by a subgroup consisting of matrices with an  $SU(2)$  matrix in the upper left corner, ones on the remaining diagonal, and zeroes elsewhere. As this subgroup also generates  $\pi_3(SU(n))$ , the map  $\alpha : \pi_3(S[U(n-l) \times U(l)]) \rightarrow \pi_3(SU(n))$  will be onto. Hence, from (2.6), we have

$$\pi_3[SU(n)/S[U(n-l) \times U(l)]] = \begin{cases} \mathbf{Z}, & \text{for } n-l = l = 1 \\ 0, & \text{for } n \geq 3, n-l \geq l \geq 1. \end{cases} \quad (4.18)$$

## B. $G = SO(n)$

Since the orthogonal groups may be thought of as unitary groups over the real numbers rather than the complex numbers, it should not be surprising that much of the structure

uncovered for the case  $G = \text{SU}(n)$  repeats itself when  $G = \text{SO}(n)$ . For example, we consider a set of  $l$  distinct  $n$ -vectors  $\phi_j^a$  ( $j = 1 \dots l$ ), all of which transform as

$$\phi_j^a \rightarrow \phi_j^{a'} = O^{a'}{}_a \phi_j^a, \quad (4.19)$$

where  $O$  is an  $n \times n$  matrix representing an element of  $\text{SO}(n)$ . If each vector attains a linearly independent vacuum expectation value, the symmetry will be spontaneously broken to  $\text{SO}(n-l)$ . In the case of  $\text{SU}(n)$  the resulting vacuum manifolds had trivial third homotopy groups (for  $n-l \geq 2$ ), because the generator of  $\pi_3(G)$  could be thought of as the generator of  $\pi_3(H)$  included in  $G$ . The same will occur for  $\text{SO}(n)$ , but only when  $n$  and  $n-l$  are sufficiently large. Consider the case  $n > n-l \geq 4$ . As usual, the inclusion  $\text{SO}(n-l) \hookrightarrow \text{SO}(n)$  places an element of  $\text{SO}(n-l)$  into the upper left corner of the  $\text{SO}(n)$  matrix. We have discovered that the generators of  $\pi_3(\text{SO}(n))$  and  $\pi_3(\text{SO}(n-l))$  are the same, and are specified by exponentiation of the Lie subalgebra with basis elements  $Z_1, Z_2, Z_3$  from (3.27). (This is not precisely true for  $n-l = 4$ , since  $\pi_3(\text{SO}(4))$  has two generators; nevertheless, the important fact is that one of the generators of  $\pi_3(\text{SO}(4))$  also generates  $\pi_3(\text{SO}(n))$ .) Thus, just as in (4.4) for the case of  $\text{SU}(n)$ , we obtain

$$p = \frac{\xi_{\text{SO}(n)}(i \circ f)}{\xi_{\text{SO}(n)}(g)} = \frac{2}{2} = 1, \quad n > n-l \geq 4, \quad (4.20)$$

leading via (3.2) to

$$\pi_3(\text{SO}(n)/\text{SO}(n-l)) = 0, \quad n > n-l \geq 4. \quad (4.21)$$

The lower-dimensional cases must be handled individually. For  $n \geq 5$  and  $n-l = 3$ , the procedure is the same, but the generator of  $\pi_3(\text{SO}(3))$  is specified by the Lie algebra (3.22). Hence  $\xi_{\text{SO}(n)}(i \circ f) = 4$ , and

$$\pi_3(\text{SO}(n)/\text{SO}(3)) = \mathbf{Z}_2, \quad n \geq 5. \quad (4.22)$$

Meanwhile, when  $n = 4$  and  $n-l = 3$ , we use  $\text{SO}(4)/\text{SO}(3) \sim S^3$  to obtain

$$\pi_3(\text{SO}(4)/\text{SO}(3)) = \mathbf{Z}. \quad (4.23)$$

The only remaining possibility is  $n - l = 2$ ; as  $\text{SO}(2)$  is abelian we have

$$\pi_3(\text{SO}(n)/\text{SO}(2)) = \pi_3(\text{SO}(n)) . \quad (4.24)$$

A set of fields  $\phi_{ab} = \phi_{ba}$  transforming in the symmetric second-rank tensor representation of  $\text{SO}(n)$  can attain a vacuum expectation value for which  $\phi_{ab}$  is diagonal; details can be found in [12]. The result is to break  $\text{SO}(n)$  to a subgroup  $H = \text{S}[\text{O}(n-l) \times \text{O}(l)] = \text{SO}(n-l) \times \text{SO}(l) \times \mathbf{Z}_2$ , where we can always choose  $n-l \geq l$ . As in the case  $H = \text{SO}(n-l)$ , the analysis is straightforward for sufficiently large  $n$ ,  $n-l$ ; using (2.9), we will ignore the  $\mathbf{Z}_2$  factor. Specifically, for  $n > n-l \geq 4$ , we have the exact sequence

$$\begin{array}{ccccc} \pi_3(\text{SO}(n)/\text{SO}(n-l)) & \longrightarrow & \pi_3(\text{SO}(n)/(\text{SO}(n-l) \times \text{SO}(l))) & \longrightarrow & \pi_2(\text{SU}(l)) \\ \parallel & & & & \parallel \\ 0 & & & & 0 \end{array} , \quad (4.25)$$

which implies

$$\pi_3[\text{SO}(n)/(\text{SO}(n-l) \times \text{SO}(l))] = 0 , \quad n > n-l \geq 4 . \quad (4.26)$$

There are only three cases for which  $n-l \leq 3$ :  $n=6, l=3$ ;  $n=5, l=2$ ; and  $n=4, l=2$ . In the last of these,  $H = \text{SO}(2) \times \text{SO}(2)$  is abelian, and we have

$$\pi_3[\text{SO}(4)/(\text{SO}(2) \times \text{SO}(2))] = \pi_3(\text{SO}(4)) = \mathbf{Z} \oplus \mathbf{Z} . \quad (4.27)$$

Similarly, for  $G = \text{SO}(5)$ ,  $H = \text{SO}(3) \times \text{SO}(2)$ , the abelian factor is irrelevant, and

$$\pi_3[\text{SO}(5)/(\text{SO}(3) \times \text{SO}(2))] = \pi_3(\text{SO}(5)/\text{SO}(3)) = \mathbf{Z}_2 . \quad (4.28)$$

Lastly, the case  $G = \text{SO}(6)$ ,  $H = \text{SO}(3) \times \text{SO}(3)$  bears a close resemblance to  $G = \text{SU}(4)$ ,  $H = \text{SO}(4)$ . From (2.6), we need only to find the image of

$$\begin{array}{ccc} \pi_3(\text{SO}(3) \times \text{SO}(3)) & \xrightarrow{\alpha} & \pi_3(\text{SO}(6)) \\ \parallel & & \parallel \\ \mathbf{Z} \oplus \mathbf{Z} & & \mathbf{Z} \end{array} . \quad (4.29)$$

The same reasoning used in the computation of  $\pi_3(\text{SU}(4)/\text{SO}(4))$  tells us that the image of  $\alpha$  is the even numbers, which implies

$$\pi_3[\text{SO}(6)/(\text{SO}(3) \times \text{SO}(3))] = \mathbf{Z}_2 . \quad (4.30)$$

The last case to consider is that of the antisymmetric second-rank tensor representation  $\phi_{ab} = -\phi_{ba}$ . As in the case of  $\text{SU}(n)$ , the scalar fields may attain a vacuum expectation value of the form (4.11), breaking  $\text{SO}(n)$  to  $\text{SO}(n-2) \times \text{SO}(2)$ ; we have computed  $\pi_3(\text{SO}(n)/(\text{SO}(n-2) \times \text{SO}(2)))$  above. The fields may also attain a vacuum expectation value of the form (4.12) if  $n = 2k$  is even, and (4.13) if  $n = 2k + 1$  is odd ( $k \geq 2$ ). In either case, the unbroken subgroup is  $H = \text{U}(k)$ . Let us consider the Lie algebra map  $i_* : \mathcal{U}(k) \rightarrow \mathcal{SO}(2k)$  corresponding to the inclusion  $i$  of  $\text{U}(k)$  into  $\text{SO}(2k)$ . It takes a complex  $k \times k$  skew-Hermitian matrix  $X$ , which we write  $X_R + iX_I$ , to a real antisymmetric  $2k \times 2k$  matrix of the form

$$i_* : X_R + iX_I \mapsto \tilde{X} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes X_R + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes X_I . \quad (4.31)$$

For  $n = 2k + 1$  the map is the same, with an extra column and row of zeroes added to  $\tilde{X}$ . Since the map  $f_* : \mathcal{U}(2) \rightarrow \mathcal{U}(k)$  is given by (3.19), it is straightforward to compute that

$$\xi_{\text{SO}(2k[+1])}(i_* \circ f_*) = 2 . \quad (4.32)$$

As the generator  $g$  of  $\pi_3(\text{SO}(n))$  satisfies  $\xi_{\text{SO}(n)}(g_*) = 2$  for all  $n \geq 5$ , we have

$$\pi_3(\text{SO}(2k[+1])/\text{U}(k)) = 0 , \quad 2k[+1] \geq 5 . \quad (4.33)$$

The only other case of interest is  $\text{SO}(4)/\text{U}(2)$ , for which precisely the same logic holds, with the exception that  $\pi_3(\text{SO}(4)) = \mathbf{Z} \oplus \mathbf{Z}$ ; therefore, the image of  $\alpha : \pi_3(\text{U}(2)) \rightarrow \pi_3(\text{SO}(4))$  is one of the  $\mathbf{Z}$  factors, and

$$\pi_3(\text{SO}(4)/\text{U}(2)) = \mathbf{Z} . \quad (4.34)$$

This completes our computation of  $\pi_3(G/H)$ .

### C. Other homotopy groups

The various symmetry breaking patterns we have studied may lead not only to textures, but also to strings, monopoles and domain walls. Here we calculate the lower homotopy groups  $\pi_1$  and  $\pi_2$  of the vacuum manifolds  $G/H$  considered above, to determine whether these theories predict strings or monopoles. (We do not study  $\pi_0$ , which governs the appearance of domain walls, as all of the groups we consider are connected, and  $\pi_0(G/H)$  is always the trivial group.) For the most part, these calculations require less effort than the computation of  $\pi_3$ .

For all Lie groups  $G$ ,  $\pi_2(G) = 0$ ; furthermore, for most of the groups we consider (those without  $Z_n$  factors),  $\pi_0(G) = 0$ . We therefore have the following exact homotopy sequence:

$$0 \xrightarrow{\beta} \pi_2(G/H) \xrightarrow{\gamma} \pi_1(H) \xrightarrow{\bar{\alpha}} \pi_1(G) \xrightarrow{\bar{\beta}} \pi_1(G/H) \xrightarrow{\bar{\gamma}} 0 . \quad (4.35)$$

For  $G = \text{SU}(n)$ , we have  $\pi_1(G) = 0$ , which leads immediately to

$$\pi_2(\text{SU}(n)/H) = \pi_1(H) , \quad (4.36)$$

and

$$\pi_1(\text{SU}(n)/H) = 0 , \quad (4.37)$$

for any choice of  $H$ .

The case  $G = \text{SO}(n)$ ,  $n \geq 3$ , requires more effort. Let's begin with  $H = \text{SO}(n-l)$ , for  $n-l \geq 3$ . We need to examine the exact sequence (4.35) with  $\pi_1(H) = \pi_1(G) = \mathbf{Z}_2$ . However, despite the fact that  $\pi_1(H)$  and  $\pi_1(G)$  are isomorphic as groups, it does not necessarily follow that the map  $\bar{\alpha}$  is itself an isomorphism. Therefore this sequence by itself is insufficient to compute the unknown homotopy groups; we must examine the map  $\bar{\alpha}$  in more detail. In a manner analogous to that discussed below (4.5) for the case of  $\text{SU}(n)$ , we may factor  $\bar{\alpha}$  into  $\bar{\alpha}_{n-1} \circ \bar{\alpha}_{n-2} \circ \dots \circ \bar{\alpha}_{n-l}$ , where  $\bar{\alpha}_m : \pi_1(\text{SO}(m)) \rightarrow \pi_1(\text{SO}(m+1))$ . We therefore consider the sequence (4.35) with  $G = \text{SO}(4)$ ,  $H = \text{SO}(3)$ . The quotient space is  $\text{SO}(4)/\text{SO}(3) \sim S^3$ , and we know that  $\pi_2(S^3) = \pi_1(S^3) = 0$ . Thus, (4.35) guarantees



that  $\bar{\alpha}_3 : \pi_1(\text{SO}(3)) \rightarrow \pi_1(\text{SO}(4))$  is an isomorphism. Similar arguments suffice to show that, for any  $m \geq 3$ ,  $\bar{\alpha}_m : \pi_1(\text{SO}(m)) \rightarrow \pi_1(\text{SO}(m+1))$  is an isomorphism, and thus that  $\bar{\alpha} : \pi_1(\text{SO}(n-l)) \rightarrow \pi_1(\text{SO}(n))$  is an isomorphism for all  $n > n-l \geq 3$ . Thus, the image of  $\bar{\alpha}$  is all of  $\pi_1(\text{SO}(n)) = \mathbf{Z}_2$ . Exactness of (4.35) implies that the kernel of  $\bar{\beta} : \mathbf{Z}_2 \rightarrow \pi_1(\text{SO}(n)/\text{SO}(n-l))$  is all of  $\mathbf{Z}_2$ , and that the image of  $\bar{\beta}$  is the kernel of  $\bar{\gamma}$ , namely all of  $\pi_1(\text{SO}(n)/\text{SO}(n-l))$ . Together these imply that

$$\pi_1(\text{SO}(n)/\text{SO}(n-l)) = 0, \quad n \geq n-l \geq 3. \quad (4.38)$$

Similar reasoning leads to

$$\pi_2(\text{SO}(n)/\text{SO}(n-l)) = 0, \quad n \geq n-l \geq 3. \quad (4.39)$$

Furthermore, we can repeat the procedure with  $H = \text{SO}(2)$ , the only modification being  $\pi_1(H) = \mathbf{Z}$ . The results in this case are

$$\pi_1(\text{SO}(n)/\text{SO}(2)) = 0, \quad n \geq 3, \quad (4.40)$$

and

$$\pi_2(\text{SO}(n)/\text{SO}(2)) = \mathbf{Z}, \quad n \geq 3. \quad (4.41)$$

For  $G/H = \text{SO}(n)/(\text{SO}(n-l) \times \text{SO}(l) \times \mathbf{Z}_2)$  we have  $\pi_0(H) = \mathbf{Z}_2$ , and hence (4.35) does not apply. Instead we may proceed in stages, first considering  $(\text{SO}(n)/\text{SO}(n-l))/\text{SO}(l)$  and then  $[\text{SO}(n)/(\text{SO}(n-l) \times \text{SO}(l))]/\mathbf{Z}_2$ . Without presenting the relevant details, it is straightforward to find that the  $\mathbf{Z}_2$  factor renders the vacuum manifold non-simply-connected:

$$\pi_1(\text{SO}(n)/(\text{SO}(n-l) \times \text{SO}(l) \times \mathbf{Z}_2)) = \mathbf{Z}_2, \quad (4.42)$$

while the second homotopy group is more complicated:

$$\pi_2(\mathrm{SO}(n)/(\mathrm{SO}(n-l) \times \mathrm{SO}(l) \times \mathbf{Z}_2)) = \begin{cases} \mathbf{Z} , & \text{for } n = 3 , l = 1 \\ 0 , & \text{for } n \geq 4 , l = 1 \\ \mathbf{Z} \oplus \mathbf{Z} , & \text{for } n = 4 , l = 2 \\ \mathbf{Z} , & \text{for } n \geq 5 , l = 2 \\ \mathbf{Z}_2 , & \text{for } n \geq 6 , l \geq 3 . \end{cases} \quad (4.43)$$

Finally, we can consider  $G = \mathrm{SO}(2k[+1])$  breaking down to  $H = \mathrm{U}(k)$ . In this case the exact sequence (4.35) by itself is insufficient; however, it is easy to check that the map  $\bar{\alpha}$  is onto, which then leads to

$$\pi_1(\mathrm{SO}(2k[+1])/\mathrm{U}(k)) = 0 , \quad (4.44)$$

and

$$\pi_2(\mathrm{SO}(2k[+1])/\mathrm{U}(k)) = \begin{cases} \mathbf{Z} , & \text{for } 2k[+1] \geq 3 \\ 0 , & \text{for } 2k[+1] = 2 . \end{cases} \quad (4.45)$$

We have tabulated the results of all of our homotopy calculations in Table Two.

## V. Discussion

We have studied the topology of the vacuum manifolds  $G/H$  resulting from the spontaneous breakdown of a global symmetry  $G$  to a subgroup  $H$ . The homotopy groups  $\pi_q(G/H)$  are related to the existence of field configurations of potential significance to cosmology: topological defects for  $q = 0, 1, 2$ , and textures for  $q = 3$ . Although it is generally straightforward to calculate  $\pi_q(G/H)$  for  $q \leq 2$ , the case  $q = 3$  relevant to textures is more difficult. We have therefore studied a number of different choices for  $G/H$ , and computed the relevant homotopy groups.

Although the results of this paper allow us to predict whether any given spontaneously broken global symmetry will lead to texture, there are a number of questions remaining to be answered about the cosmological effects of those textures which result. For example, one may ask what the likelihood is that a specified field configuration will collapse and unwind, and how many such unwindings per horizon volume are predicted by the Kibble

mechanism. In the same vein, it would be of interest to know the characteristics of collapse; *e.g.*, whether the field approaches a spherically symmetric configuration, rather than a pancake or spindle configuration. All of these issues may in principle depend on the geometry of the vacuum manifold  $G/H$  under consideration. Hence, both analytic and numerical studies of the evolution of textures resulting from different choices of  $G$  and  $H$  would be of interest. (Such studies have been performed in the case of  $G/H = S^3$  [14] and  $G/H = S^2$  [15].)

While we will not attempt to answer any of these questions in this paper, we would like to briefly discuss the construction of field configurations representing individual textures, which would be appropriate initial conditions for simulations. We imagine therefore that we have a set of scalar fields  $\Phi$  which transform under some representation of the global symmetry group  $G$ . We will write the action of an element  $\mu \in G$  on  $\Phi$  as  $\mu\Phi$ , although the explicit matrix form may be more complicated. Our goal is to specify a field configuration  $\Phi(x)$  which is in the vacuum manifold (or, equivalently, in  $G/H$ ) and has a given winding number.

The action of  $G$  on the vacuum manifold is transitive — a fixed element is taken to any other element by the action of the group. Therefore, every field value  $\Phi$  in the vacuum manifold is of the form  $\mu\Phi_0$ , where we have specified a fiduciary field value  $\Phi_0$  in the vacuum manifold. Hence, our sought-after field configuration  $\Phi(x)$  can be written

$$\Phi(x) = \mu(x)\Phi_0 , \tag{5.1}$$

where  $\mu(x)$  is a map from space (at fixed time) to the symmetry group  $G$ . (Such a description may be highly redundant, as two different group elements  $\mu_1$  and  $\mu_2$  may satisfy  $\mu_1\Phi_0 = \mu_2\Phi_0$ ; however, this redundancy is not a concern in the construction of a field configuration.) We are only considering maps  $\mu(x)$  which go to a constant element of  $G$  at spatial infinity; that is,  $\mu(x)$  represents a map  $S^3 \rightarrow G$ , and hence an element of  $\pi_3(G)$ . Acting  $\mu(x)$  on  $\Phi_0$  therefore produces a map  $S^3 \rightarrow G/H$ , and hence an element of  $\pi_3(G/H)$ ; indeed, we have just exhibited the map  $\beta : \pi_3(G) \rightarrow \pi_3(G/H)$  found in the exact homotopy sequence.

Since we would like  $\Phi(x)$  to represent a nonzero element of  $\pi_3(G/H)$ , we are interested in elements of  $\pi_3(G)$  which are not in the kernel of  $\beta$ . This is straightforward, given the results of the previous sections, which allow us to construct the map  $\beta$  explicitly by taking advantage of the fact that the kernel of  $\beta$  was equal to the image of  $\alpha : \pi_3(H) \rightarrow \pi_3(G)$ . For example, consider the case  $G = \text{SU}(3)$ ,  $H = \text{SO}(3)$ , for which  $\pi_3(G) = \mathbf{Z}$  and  $\pi_3(G/H) = \mathbf{Z}_4$ , and  $\beta$  is “mod 4.” Thus, elements of  $\pi_3(\text{SU}(3))$  with winding numbers 1, 2, or 3

will be taken by  $\beta$  to maps with the same winding number in  $\pi_3(\text{SU}(3)/\text{SO}(3))$ , while a map with winding number 4 will be taken to a map with winding number zero, and so on. To specify a nonzero element of  $\pi_3(\text{SU}(3)/\text{SO}(3))$ , we therefore need only to find a map  $\mu(x)$  representing  $\pi_3(\text{SU}(3))$  with winding number one (for example). This is also straightforward, as we have specified the generators of  $\pi_3(G)$  in Sec. III.B, in terms of group homomorphisms  $\text{SU}(2) \rightarrow G$ . Thus, we specify a map  $\tilde{\mu}(x)$  from  $S^3$  (representing space) to  $\text{SU}(2)$  which generates  $\pi_3(\text{SU}(2))$ , and a map  $g : \text{SU}(2) \rightarrow G$  which generates  $\pi_3(G)$ ; the field configuration  $\Phi = \mu(x)\Phi$ , where  $\mu(x) = g \circ \tilde{\mu}(x)$ , will then have winding number one (as long as  $\beta$  takes a generator of  $\pi_3(G)$  to a generator of  $\pi_3(G/H)$ ).

Let us illustrate this procedure for  $G = \text{SU}(3)$ ,  $H = \text{SO}(3)$ . In a Cartesian coordinate system  $(x, y, z)$ , an example of a map from space to  $\text{SU}(2)$  which covers the group once (and hence generates  $\pi_3(\text{SU}(2))$ ) is given by

$$\tilde{\mu}(x) = \mathbf{I} \cos \chi(r) - i \vec{\sigma} \cdot \hat{x} \sin \chi(r) , \quad (5.2)$$

where  $\mathbf{I}$  is a  $2 \times 2$  identity matrix, the  $\sigma_i$  are the Pauli matrices (related to the Lie algebra elements  $X_i$  of (3.16) by  $\sigma_i = -2iX_i$ ),  $r = \sqrt{x^2 + y^2 + z^2}$ ,  $\hat{x}_i = x_i/r$ , and  $\chi(r)$  is a function with boundary conditions  $\chi(0) = 0$  and  $\chi(\infty) = \pi$ . In polar coordinates  $(r, \theta, \phi)$  this becomes

$$\tilde{\mu}(x) = \begin{pmatrix} \cos \chi - i \cos \theta \sin \chi & e^{i\phi} \sin \theta \sin \chi \\ -e^{-i\phi} \sin \theta \sin \chi & \cos \chi + i \cos \theta \sin \chi \end{pmatrix} . \quad (5.3)$$

As usual, we will pick the generator of  $\pi_3(\text{SU}(3))$  to be represented by the inclusion  $g : \text{SU}(2) \rightarrow \text{SU}(3)$  in the upper left corner. The scalar fields which break  $\text{SU}(3)$  to  $\text{SO}(3)$  lie in the symmetric second-rank tensor representation of  $\text{SU}(3)$ , and attain a vacuum expectation value of the form  $\langle \phi_{ab} \rangle = \rho \delta_{ab}$ , with  $\rho$  an arbitrary constant. It is natural to choose as our fiducial value of the field  $\Phi_0 = \langle \phi_{ab} \rangle$ ; then, using the transformation law (4.6), the field configuration becomes

$$\phi_{ab}(x) = \rho U^c_a(x) U^d_b(x) \delta_{cd} , \quad (5.4)$$

where  $U^c_a(x)$  is an  $\text{SU}(3)$  matrix representing  $\mu(x) = g \circ \tilde{\mu}(x)$ . Using (5.3), we therefore

have

$$\phi_{ab}(x) = \rho \begin{pmatrix} 1 - 2ic_\theta c_\chi s_\chi - 2s_\chi^2 + (1 + e^{-2i\phi})s_\theta^2 s_\chi^2 & 2i(s_\phi c_\chi - c_\phi c_\theta s_\chi)s_\theta s_\chi & 0 \\ 2i(s_\phi c_\chi - c_\phi c_\theta s_\chi)s_\theta s_\chi & 1 + 2ic_\theta c_\chi s_\chi - 2s_\chi^2 + (1 + e^{2i\phi})s_\theta^2 s_\chi^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.5)$$

where  $c_\chi \equiv \cos \chi$  and so on. This configuration represents a texture of winding number one, the evolution of which could be studied numerically.

From this point, it is easy to construct additional configurations with the same winding number, simply by choosing different maps  $\mu : S^3 \rightarrow G$  which generate  $\pi_3(G)$ . For example, for any submanifold  $\Sigma$  of  $G$ , conjugation by a fixed element  $m$  defines a new submanifold  $m\Sigma m^{-1}$  homotopic to  $\Sigma$ ; hence, if  $\mu(x)$  generates  $\pi_3(G)$ , so will  $\mu'(x) = m\mu(x)m^{-1}$ . Furthermore, given two maps  $\mu_1(x)$  and  $\mu_2(x)$ , the winding number in  $\pi_3(G)$  obeys

$$\eta_G(\mu_1\mu_2) = \eta_G(\mu_1) + \eta_G(\mu_2). \quad (5.6)$$

Therefore, field configurations with higher winding numbers are readily constructed (if they exist at all). These could be single textures with winding number greater than one, or two nearby textures.

Although it is still unclear whether topological properties of spontaneously broken symmetries play a role in the formation of large-scale structure in the universe, the lack of a single compelling model of structure formation encourages further study of many different models. The calculations performed in this paper provide a starting point for the study of a number of models beyond those considered to date; further work should enable us to determine the relationship of these theories to the observed universe.

## Acknowledgements

We would like to thank Larry Widrow for encouragement, Alan Guth and Andrew Sornborger for useful comments, and especially Sidney Coleman for invaluable help with our  $\pi$ 's. This work was supported in part by NASA under Grants no. NAGW-931 and NGT-50850, by the National Science Foundation under grants AST/90-05038 and PHY/92-06867, and by the U.S. Department of Energy (D.O.E.) under contract no. DE-AC02-76ER03069.

## Appendix: Factorization

Whenever we have two spaces  $X$  and  $Y$  and a map  $\phi : X \rightarrow Y$ , there is an induced map in homotopy  $\Phi : \pi_q(X) \rightarrow \pi_q(Y)$ . In calculating the action of  $\Phi$ , we are often aided by the existence of a third space  $Z$  in between  $X$  and  $Y$ , in the sense that there are maps  $\psi_1 : X \rightarrow Z$  and  $\psi_2 : Z \rightarrow Y$  (which induce maps  $\Psi_1$  and  $\Psi_2$  in homotopy) such that  $\phi = \psi_2 \circ \psi_1$ . In that case, we can factor the map  $\Phi$  into  $\Psi_2 \circ \Psi_1$ . In other words, if the diagram between topological spaces

$$\begin{array}{ccc} X & \xrightarrow{\psi_1} & Z \\ \phi \searrow & & \downarrow \psi_2 \\ & & Y \end{array} \quad (A.1)$$

commutes, then the diagram between homotopy groups

$$\begin{array}{ccc} \pi_q(X) & \xrightarrow{\Psi_1} & \pi_q(Z) \\ \Phi \searrow & & \downarrow \Psi_2 \\ & & \pi_q(Y) \end{array} \quad (A.2)$$

will also commute.<sup>5</sup> (Recall that a diagram is said to commute if, for any two objects in the diagram and any two maps between the objects, obtained by composition of maps in the diagram, those two maps coincide.) This is straightforward to show, by considering how the maps between spaces induce maps between their homotopy groups.

Given  $\phi : X \rightarrow Y$ , we construct  $\Phi : \pi_q(X) \rightarrow \pi_q(Y)$  in the following way. Fix a map  $f_0 : S^q \rightarrow X$ , which represents the homotopy class  $[f_0] \in \pi_q(X)$ . Then  $\phi \circ f_0$  is a map from  $S^q \rightarrow Y$ , which represents  $[\phi \circ f_0] \in \pi_q(Y)$ . We therefore define  $\Phi$  via  $\Phi : [f_0] \mapsto [\phi \circ f_0]$ . We need merely to show that this definition is independent of our choice of  $f_0$ , *i.e.* that it sends two homotopic maps  $f_0, f_1 : S^q \rightarrow X$  to the same class in  $\pi_q(Y)$ . To do this, consider a homotopy<sup>4</sup> from  $f_0$  to  $f_1$ , given by a map  $F : [0, 1] \times S^q \rightarrow X$  which satisfies  $F(0) = f_0, F(1) = f_1$ . Then the map  $\phi \circ F$  serves as a homotopy from  $\phi \circ f_0$  to  $\phi \circ f_1$ , and hence the map  $\Phi$  is well defined on homotopy classes.

Now consider the composition  $\Psi_2 \circ \Psi_1$ .  $\Psi_1$  takes the homotopy class of  $f : S^q \rightarrow X$  and maps it to the homotopy class of  $\psi_1 \circ f : S^q \rightarrow Z$ , while  $\Psi_2$  takes the homotopy class of  $\tilde{f} : S^q \rightarrow Z$  to the homotopy class of  $\psi_2 \circ \tilde{f} : S^q \rightarrow Y$ . If we choose  $\tilde{f} = \psi_1 \circ f$ , we find that  $\Psi_2 \circ \Psi_1 : [f] \mapsto [\psi_2 \circ (\psi_1 \circ f)]$ . Since composition is associative, we have shown that  $\Psi_2 \circ \Psi_1$  is the same map as that induced by  $\psi_2 \circ \psi_1$ . Thus, if  $\phi = \psi_2 \circ \psi_1$ , then  $\Phi = \Psi_2 \circ \Psi_1$ .

---

<sup>5</sup> In still other words, we are defining the homotopy functor from the category of topological spaces to the category of groups. See [16].

<sup>4</sup> Each map must have the same base point. That is,  $f_0, f_1$  and  $F$  must all send the north pole of  $S^q$  to the same point in  $X$ .

## REFERENCES

1. N. Turok, *Phys. Rev. Lett.* **63**, 2625 (1989).
2. N. Turok and D. Spergel, *Phys. Rev. Lett.* **64**, 2763 (1990); D. Spergel, N. Turok, W.H. Press, and B.S. Ryden, *Phys. Rev. D* **43**, 1038 (1991); R.Y. Cen, J.P. Ostriker, D.N. Spergel, and N. Turok, *Ap. J* **393**, 42 (1992).
3. D.P. Bennett and S.H. Rhie, *Ap. J. Lett.* **406**, 7 (1993); U. Pen, D.N. Spergel, and N. Turok, *Phys. Rev. D*, in press (1993).
4. I. Chuang, R. Durrer, N. Turok, and B. Yurke, *Science* **251**, 1336 (1991).
5. T. Skyrme, *Proc. Roy. Soc.* **A262**, 237 (1961).
6. R. Holman, S.D.H. Hsu, E.W. Kolb, R. Watkins, and L.M. Widrow, *Phys. Rev. Lett* **69**, 1489 (1992); M. Kamionkowski and J. March-Russell, *Phys. Rev. Lett.* **69**, 1485 (1992).
7. L. Perivolaropoulos, *Phys. Rev. D* **46**, 1858 (1992).
8. M.J. Greenberg and J.R. Harper *Algebraic Topology* (Benjamin/Cummings Co., Reading, MA, 1981).
9. G.W. Whitehead, *Elements of Homotopy Theory* (Springer-Verlag, New York, 1978).
10. S. Coleman, *Aspects of Symmetry* (Cambridge Univ. Press, Cambridge, UK, 1985).
11. R. Bott, *Bull. Soc. Math. France* **84**, 251 (1956).
12. L.-F. Li, *Phys. Rev D* **9**, 1723 (1974).
13. E.J. Weinberg, D. London, and J.L. Rosner, *Nucl. Phys.* **B236**, 90 (1984).
14. J. Borrill, E.J. Copeland, and A.R. Liddle, *Phys. Lett.* **B258**, 310 (1991); R.A. Leese and T. Prokopec, *Phys. Rev. D* **44**, 3749 (1991); T. Prokopec, A. Sornborger, and R.H. Brandenberger, *Phys. Rev. D* **45**, 1971 (1992); S. Aminneborg, *Nucl. Phys.* **B388**, 521 (1992).
15. S.H. Rhie and D.P. Bennett, preprint UCRL-JC-110560 (1992); X. Luo, *Phys. Lett.* **B287**, 319 (1992).
16. R. Geroch, *Mathematical Physics* (Univ. Chicago Press, Chicago, 1985).

### Table Captions.

Table One. We list the homotopy groups  $\pi_1$  through  $\pi_3$  for the spheres and compact Lie groups. “Ex. Groups” refers to the exceptional groups  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ , and  $G_2$ .

Table Two. We list the results of our computation of the homotopy groups of vacuum manifolds  $G/H$  for various choices of  $G$  and  $H$ , as well as the dimensionality of  $G/H$ . The integer  $n - l$  is always taken to be greater than or equal to  $l$ .



**Table I: Homotopy of Lie Groups and Spheres**

Space $X$	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$
$S^1$	$\mathbf{Z}$	0	0
$S^2$	0	$\mathbf{Z}$	$\mathbf{Z}$
$S^3$	0	0	$\mathbf{Z}$
$S^{n \geq 4}$	0	0	0
$\mathrm{SO}(3)$	$\mathbf{Z}_2$	0	$\mathbf{Z}$
$\mathrm{SO}(4)$	$\mathbf{Z}_2$	0	$\mathbf{Z} \oplus \mathbf{Z}$
$\mathrm{SO}(n \geq 5)$	$\mathbf{Z}_2$	0	$\mathbf{Z}$
$\mathrm{SU}(n \geq 2)$	0	0	$\mathbf{Z}$
$\mathrm{S}[\mathrm{U}(1) \times \mathrm{U}(1)]$	$\mathbf{Z}$	0	0
$\mathrm{S}[\mathrm{U}(n) \times \mathrm{U}(1)]$	$\mathbf{Z}$	0	$\mathbf{Z}$
$\mathrm{S}[\mathrm{U}(n) \times \mathrm{U}(m)]$	$\mathbf{Z}$	0	$\mathbf{Z} \oplus \mathbf{Z}$
$\mathrm{Sp}(n \geq 1)$	0	0	$\mathbf{Z}$
Ex. Groups	0	0	$\mathbf{Z}$

**Table 2: Homotopy of Vacuum Manifolds**

$G$	$H$	$\pi_1(G/H)$	$\pi_2(G/H)$	$\pi_3(G/H)$	$\dim(G/H)$
$SU(n)$	0	0	0	$\mathbf{Z}$	$n^2 - 1$
$SU(2)$	$U(1), SO(2)$	0	$\mathbf{Z}$	$\mathbf{Z}$	2
$SU(n)$	$SU(n-2) \times SU(2)$	0	0	0	$4n - 7$
$SU(n \geq 3)$	$S[U(n-l) \times U(l)]$	0	$\mathbf{Z}$	0	$2l(n-l)$
$SU(n \geq 3)$	$SU(n-l)$	0	0	0	$l(2n-l)$
$SU(2l[+1])$	$Sp(2l)$	0	0	0	$2l^2 - l - 1 [+4l + 1]$
$SU(3)$	$SO(3)$	0	$\mathbf{Z}_2$	$\mathbf{Z}_4$	5
$SU(n \geq 4)$	$SO(n)$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\frac{1}{2}n(n+1) - 1$
$SO(n \neq 4)$	0	$\mathbf{Z}_2$	0	$\mathbf{Z}$	$\frac{1}{2}n(n-1)$
$SO(4)$	0	$\mathbf{Z}_2$	0	$\mathbf{Z} \oplus \mathbf{Z}$	6
$SO(n \neq 4)$	$SO(2)$	0	$\mathbf{Z}$	$\mathbf{Z}$	$\frac{1}{2}n(n-1) - 1$
$SO(4)$	$SO(2)$	0	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	5
$SO(4)$	$SO(3)$	0	0	$\mathbf{Z}$	3
$SO(n \geq 5)$	$SO(3)$	0	0	$\mathbf{Z}_2$	$\frac{1}{2}n(n-1) - 3$
$SO(n \geq 5)$	$SO(n-l)$	0	0	0	$\frac{1}{2}l(2n-l-1)$
$SO(3)$	$SO(2) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$	$\mathbf{Z}$	2
$SO(4)$	$SO(3) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	3
$SO(n \geq 5)$	$SO(n-1) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	0	0	$n - 1$
$SO(4)$	$SO(2) \times SO(2) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	4
$SO(5)$	$SO(3) \times SO(2) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$	$\mathbf{Z}_2$	6
$SO(n \geq 6)$	$SO(n-2) \times SO(2) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$	0	$2(n-2)$
$SO(6)$	$SO(3) \times SO(3) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	9
$SO(n \geq 7)$	$SO(n-l) \times SO(l) \times \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$l(n-l)$
$SO(4)$	$U(2)$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	2
$SO(2k[+1])$	$U(k)$	$\mathbf{Z}_2$	0	0	$k^2 - k [+2k]$